Problem 2.9

We solved the differential equation (2.29), $m\dot{v}_y = -b(v_y - v_{\text{ter}})$, for the velocity of an object falling through air, by inspection — a most respectable way of solving differential equations. Nevertheless, one would sometimes like a more systematic method, and here is one. Rewrite the equation in the "separated" form

$$\frac{m\,dv_y}{v_y - v_{\text{ter}}} = -b\,dt$$

and integrate both sides from time 0 to t to find v_y as a function of t. Compare with (2.30).

Solution

Equation (2.29) is on page 51.

$$m\dot{v}_y = -b(v_y - v_{\text{ter}})$$

$$m\frac{dv_y}{dt} = -b(v_y - v_{\text{ter}})$$

Separate variables.

$$\frac{dv_y}{v_y - v_{\text{ter}}} = -\frac{b}{m}\,dt$$

Integrate both sides definitely, assuming that at $t = 0$ the particle has velocity $v_{y\text{o}}$ and at some time of interest t the particle has velocity v_y. Because the integration is definite, no constant of integration is needed.

$$\int_{v_{y\text{o}}}^{v_y} \frac{dv'_y}{v'_y - v_{\text{ter}}} = \int_0^t -\frac{b}{m}\,dt'$$

$$\ln|v'_y - v_{\text{ter}}|\Big|_{v_{y\text{o}}}^{v_y} = -\frac{b}{m}t$$

$$\ln|v_y - v_{\text{ter}}| - \ln|v_{y\text{o}} - v_{\text{ter}}| = -\frac{b}{m}t$$

$$\ln\left|\frac{v_y - v_{\text{ter}}}{v_{y\text{o}} - v_{\text{ter}}}\right| = -\frac{b}{m}t$$

Exponentiate both sides.

$$\left|\frac{v_y - v_{\text{ter}}}{v_{y\text{o}} - v_{\text{ter}}}\right| = e^{-bt/m}$$

Remove the absolute value on the left by placing $\pm$ on the right.

$$\frac{v_y - v_{\text{ter}}}{v_{y\text{o}} - v_{\text{ter}}} = \pm e^{-bt/m}$$

Since $v_y = v_{y\text{o}}$ at $t = 0$, discard the minus sign.

$$\frac{v_y - v_{\text{ter}}}{v_{y\text{o}} - v_{\text{ter}}} = e^{-bt/m}$$

Multiply both sides by $v_{y\text{o}} - v_{\text{ter}}$.

$$v_y - v_{\text{ter}} = (v_{y\text{o}} - v_{\text{ter}})e^{-bt/m}$$

Therefore,

$$v_y(t) = v_{\text{ter}} + (v_{y\text{o}} - v_{\text{ter}})e^{-bt/m}.$$

This is Equation (2.30) on page 51 with $\tau = m/b$.